Decision Procedures over Sophisticated Fractional Permissions

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National University of Singapore

What are Fractional Permissions?

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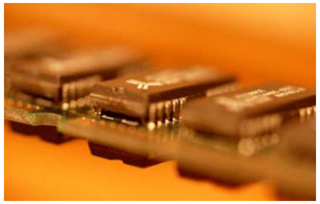
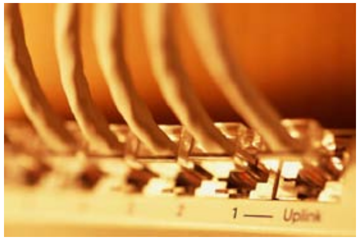
- Resource: memory cell, file on disk, network connection…





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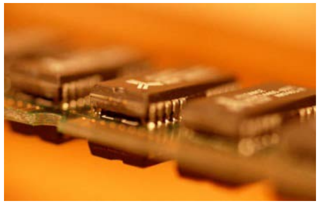
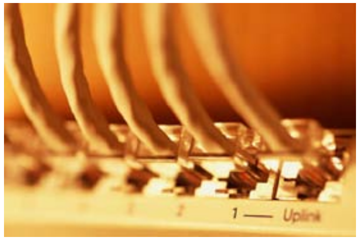
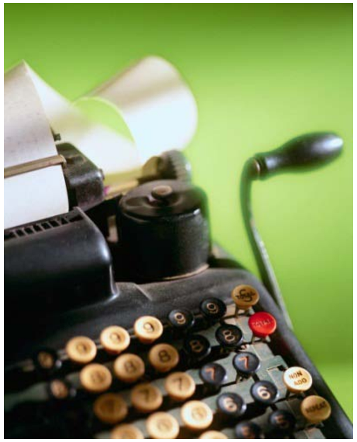




- Shared control: usually between two or more parallel computations

Accounting



- How we keep track of who owns **how much**
  - e.g., a **share** is a rational in [0, 1]

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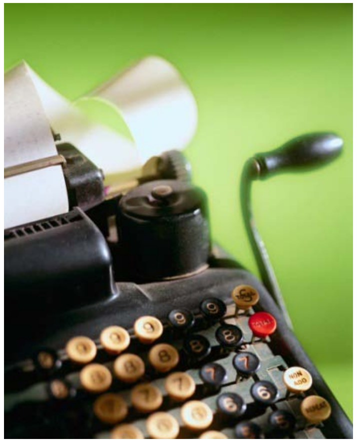

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  - we combine shares using partial addition, i.e.
    0.25 + 0.25 = 0.5   but   0.75 + 0.75 is undefined

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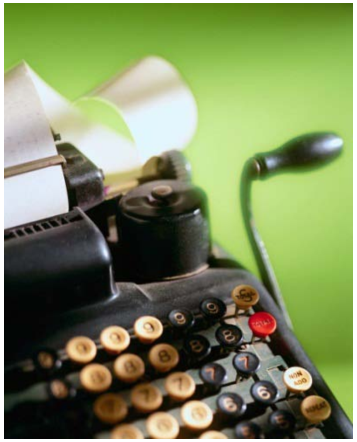

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- Not the same as **policy**, which maps shares to behaviors:
  - {1} : can write to memory cell
  - (0,1] : can read from memory cell
  - {0} : cannot use memory cell

Tree shares

- Rationals do not satisfy exactly the “right” axioms
  - See Parkinson’s thesis or our APLAS 09 paper for why.

[R. Dockins, A. Hobor, A. W. Appel. A Fresh Look at Separation Algebras and Share Accounting, APLAS 2009]

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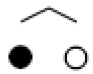
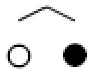
  - Left half: 
  - Right half: 

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These are not the same half share!

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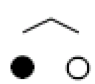
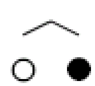
  - Left half: 
  - Right half: 
  - First quarter: 
  - etc.

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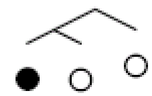
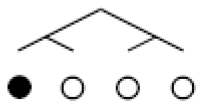
Canonical Forms

- Note we wrote the first quarter as  instead of . This is deliberate; the second is not in **canonical form**, which ensures unique representations.

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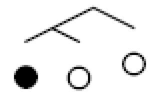
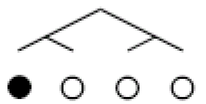
- Note we wrote the first quarter as  instead of . This is deliberate; the second is not in **canonical form**, which ensures unique representations.

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- Define a reflexive, transitive relation $\cong$ from:

$$\frac{}{\circ \cong \widehat{\circ\ \circ}} \qquad \frac{}{\bullet \cong \widehat{\bullet\ \bullet}}$$

- A tree is in canonical form when it is in the most compact representation under $\cong$.

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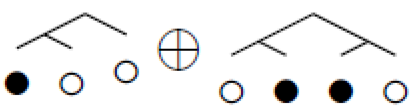
Addition

- To add trees (a partial operation), we



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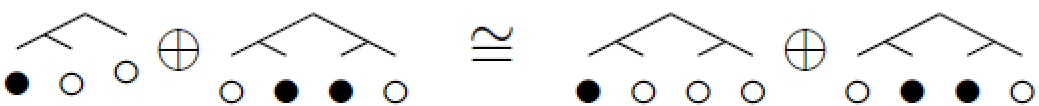

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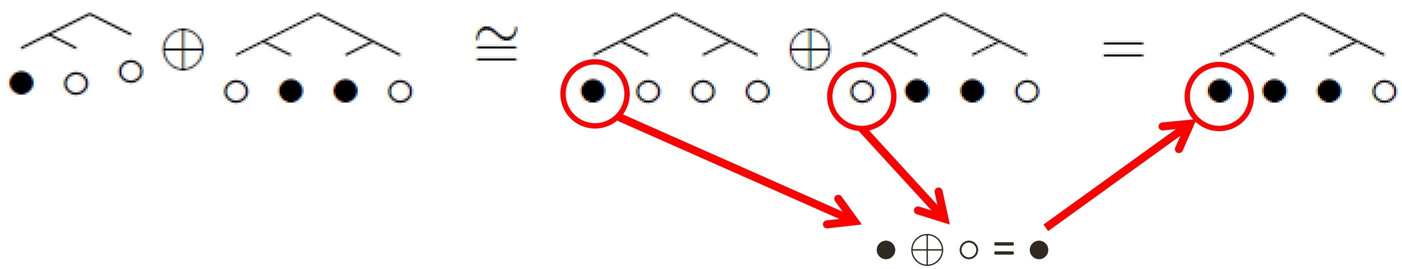

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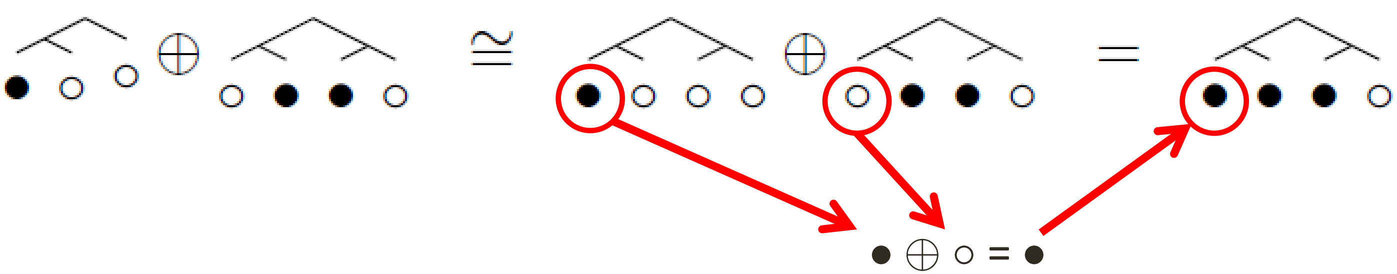

Emphasis: $\bullet \oplus \bullet$ is undefined!

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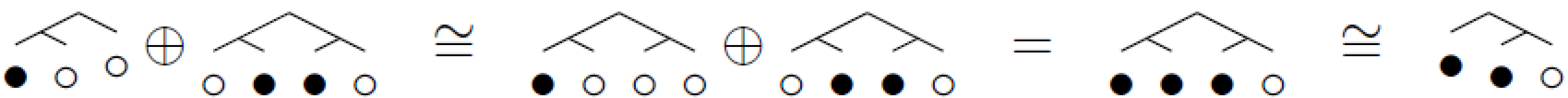
  3. Re-canonicalize



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Using shares in separation logic

- Update “maps-to” to take a tree-share:
  - $e \overset{\pi}{\mapsto} e'$
  - the current heap has a single cell e, whose value is e’, and which is owned with tree-fraction π

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5: $\boxed{7}$ * 8: $\boxed{5}$ * 5: $\boxed{7}$ = False

These shares cannot be added together!

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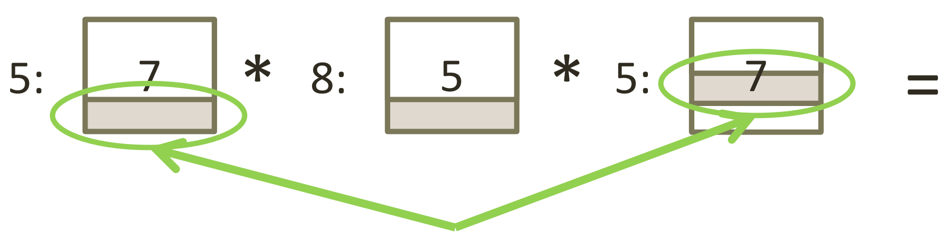

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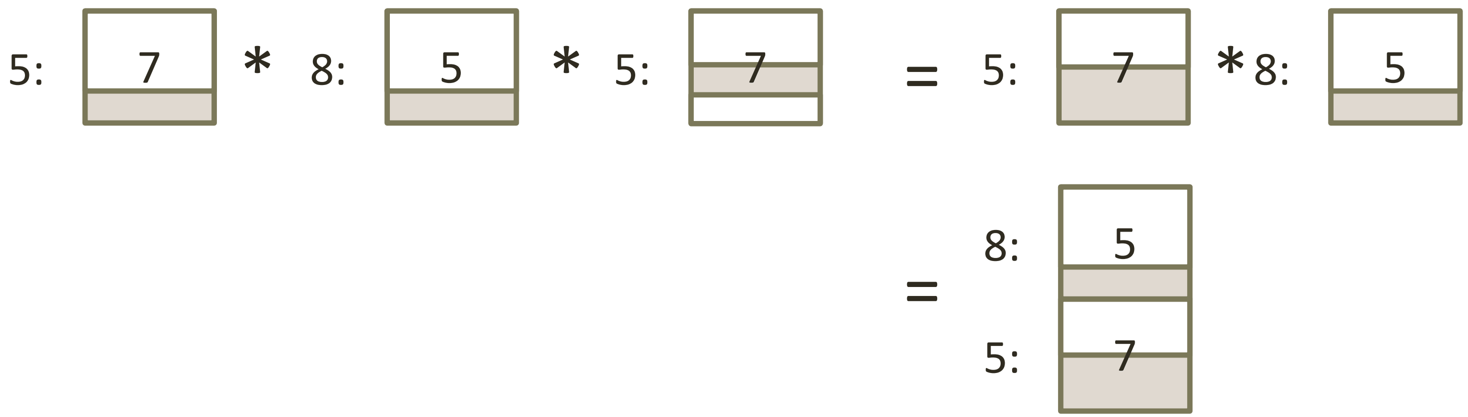

Plan of attack

1. Fractional Shares ✔
2. Verification Tools
3. Our Decision Procedures
4. Completeness
5. Experimental Results

Verification tools

- Once you have a good share model, and have integrated it into a program logic, you would like to use the logic to prove programs.

[H. H. Nguyen, C. David, S. Qin, W. N. Chin. Automated verification of shape and size properties via separation logic. VMCAI 2007]

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- We have modified the HIP/SLEEK toolchain to verify programs using fractional permissions.

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Actually, modifying SLEEK is not the major difficulty…

- SLEEK (and many other toolchains) maintains a stable of backend provers for specific domains.
  - Omega (Presburger arithmetic)
  - MONA (bags, etc.)
  - Redlog (real arithmetic)
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[A. Hobor, C. Gherghina. Barriers in concurrent separation logic: now with tool support! LMCS Vol. 8, 2011]

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- We fit into this pattern: **our major accomplishment is a backend prover for tree-shares. Our prover should be re-usable (as a library or standalone) in many other toolchains.**

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SLEEK's job

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  - which is satisfied by a decidable equality check: $\widehat{\bullet\ \circ} \overset{?}{=} \widehat{\circ\ \bullet}$ (false).

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Formal statement of problem

- SLEEK outputs systems of share equations:
  - $\phi \equiv \exists v.\ \phi$
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These are the share constants, like $\widehat{\bullet\ \circ}$

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SLEEK does not need to know much about the underlying domain of tree-shares to isolate the associated facts

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This output format is a useful modularity boundary we discovered by experimentation

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  - (IMPL) Does one system of equations imply another?

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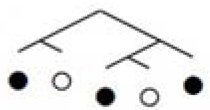
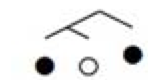
- Surprisingly, this intution is wrong: we do a **shape-guided finite search**, armed with some **completeness results** that say our finite search is sufficient.

Decomposition (SAT)

- We want to know if the following system is satisfiable:
  - x $\oplus$  = y $\wedge$ y $\oplus$ z = 

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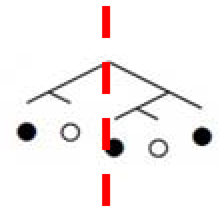
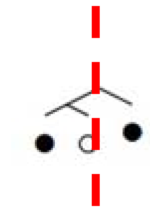
  - $x \oplus$  $= y \quad \wedge \quad y \oplus z =$ 
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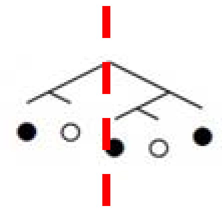
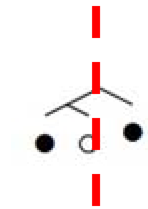
  - $x \oplus$  $= y \quad \wedge \quad y \oplus z =$ 
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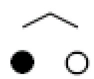
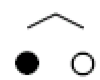
1. $x_l \oplus$  $= y_l \quad \wedge \quad y_l \oplus z_l =$ 
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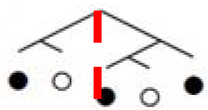
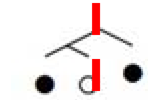
  - $x \oplus$  $= y \quad \wedge \quad y \oplus z =$ 
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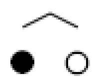
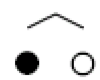
1. $x_l \oplus$  $= y_l \quad \wedge \quad y_l \oplus z_l =$ 
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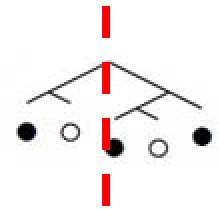
  - $x \oplus$  $= y \quad \wedge \quad y \oplus z =$ 
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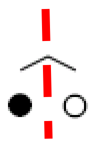
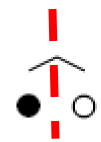
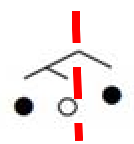
1. $x_l \oplus$  $= y_l \quad \wedge \quad y_l \oplus z_l =$ 
2. $x_r \oplus$  $= y_r \quad \wedge \quad y_r \oplus z_r =$ 

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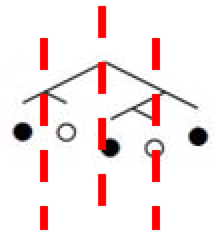
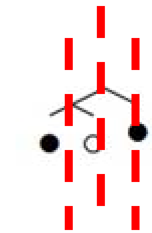
  - $x \oplus$  $= y \quad \wedge \quad y \oplus z =$ 
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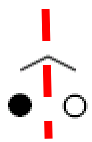
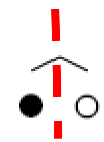
1. $x_l \oplus$  $= y_l \quad \wedge \quad y_l \oplus z_l =$ 
   a) $x_{ll} \oplus \bullet = y_{ll} \quad \wedge \quad y_{ll} \oplus z_{ll} = \bullet$
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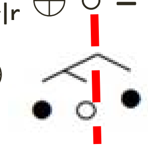
2. $x_r \oplus$  $= y_r \quad \wedge \quad y_r \oplus z_r =$ 
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Once every constant is ● or ○…

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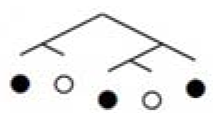
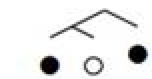
- Then we hand this problem to an SMT solver (e.g. Z3)

Decomposition (IMPL)

- The procedure for IMPL is very similar, but we have to decompose across the entailment:

  - From $x \oplus$  $= y \quad \vdash \quad \exists z.\ y \oplus z =$ 

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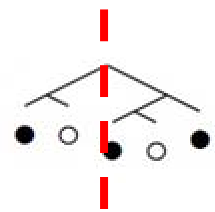
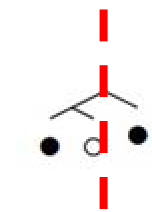
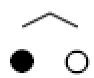
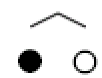
  - From $x \oplus$  $= y \quad \vdash \quad \exists z.\ y \oplus z =$ 

    1. $x_l \oplus$  $= y_l \quad \vdash \quad \exists z_l.\ y_l \oplus z_l =$ 
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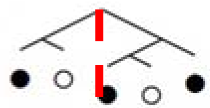
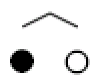
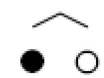
  - From $x \oplus$  $= y \quad \vdash \quad \exists z.\ y \oplus z =$ 

    1. $x_l \oplus$  $= y_l \quad \vdash \quad \exists z_l.\ y_l \oplus z_l =$ 
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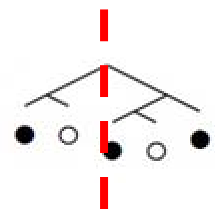
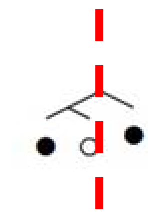
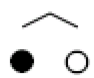
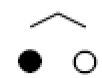
  - From $x \oplus$  $= y \quad \vdash \quad \exists z.\ y \oplus z =$ 

    1. $x_l \oplus$  $= y_l \quad \vdash \quad \exists z_l.\ y_l \oplus z_l =$ 
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- Once we have reached height zero, we apply a more complicated completeness theorem and then again translate to Boolean SAT for Z3.

Plan of attack

1. Fractional Shares ✔
2. Verification Tools ✔
3. Our Decision Procedures ✔
4. Completeness
5. Experimental Results

Completeness theorem (SAT)

- Theorem 1: finite search for SAT
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Height: highest tree-constant contained in σ or Σ.

A solution (map from tree variables to tree constants)

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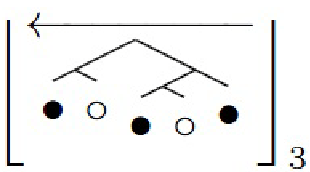
  - Proofs by example: properties of rounding
  - Proof sketch of main theorem

Tree rounding

- Define $\lfloor \overleftarrow{\tau} \rfloor_n$ “left round tree τ to height n” as follows:



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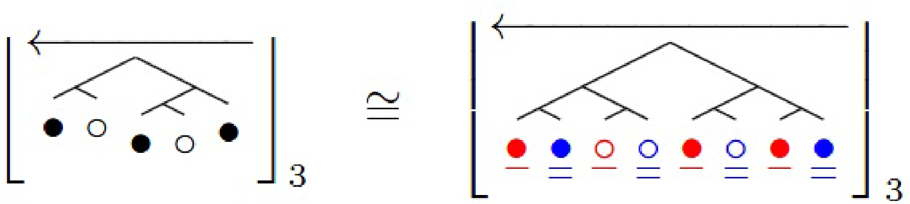

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- Define $\lfloor \overleftarrow{\tau} \rfloor_n$ "left round tree τ to height n" as follows:
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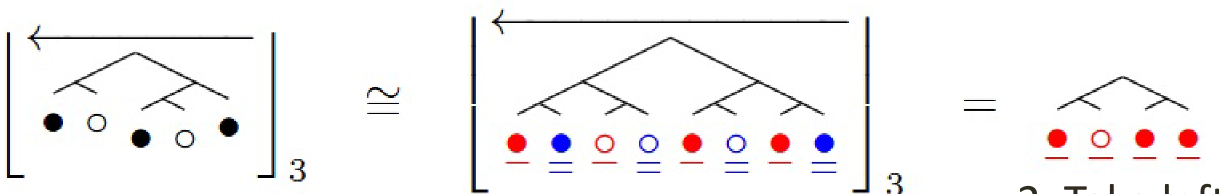

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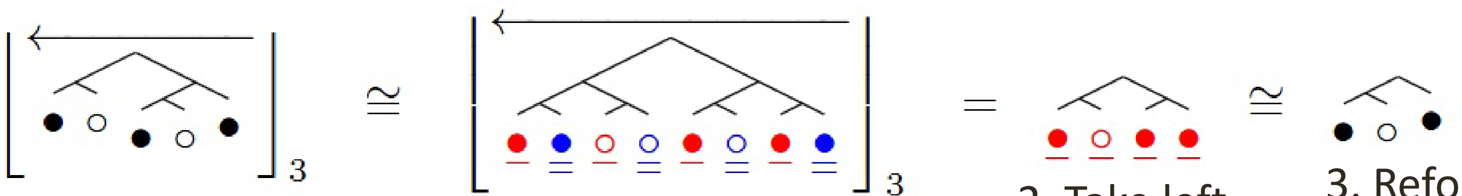

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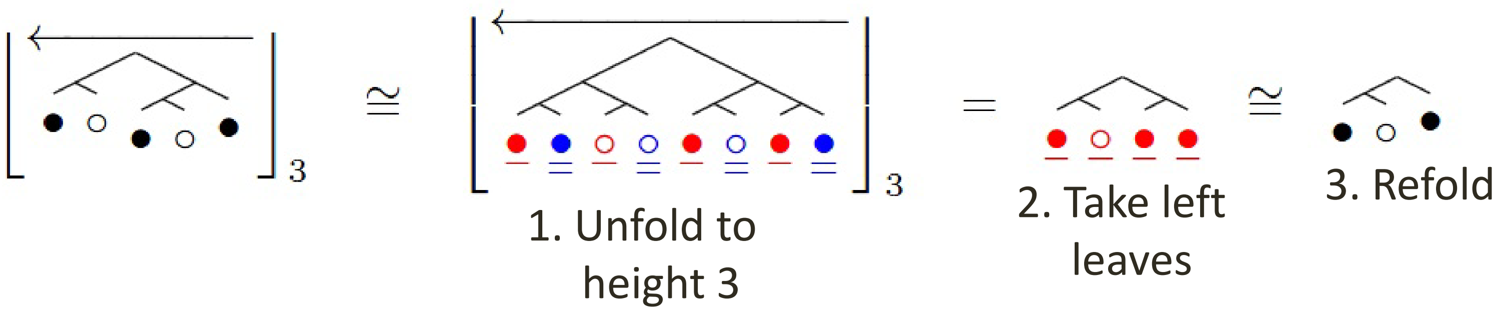
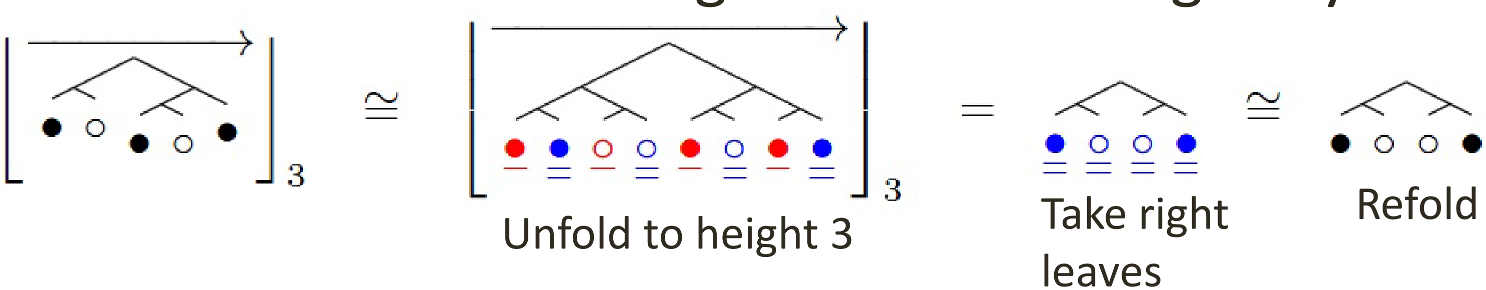


- We can also define “right round” analogously:



Key properties of rounding

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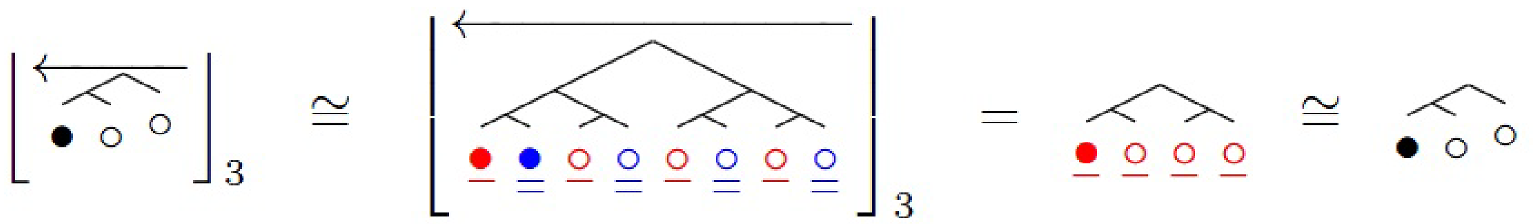

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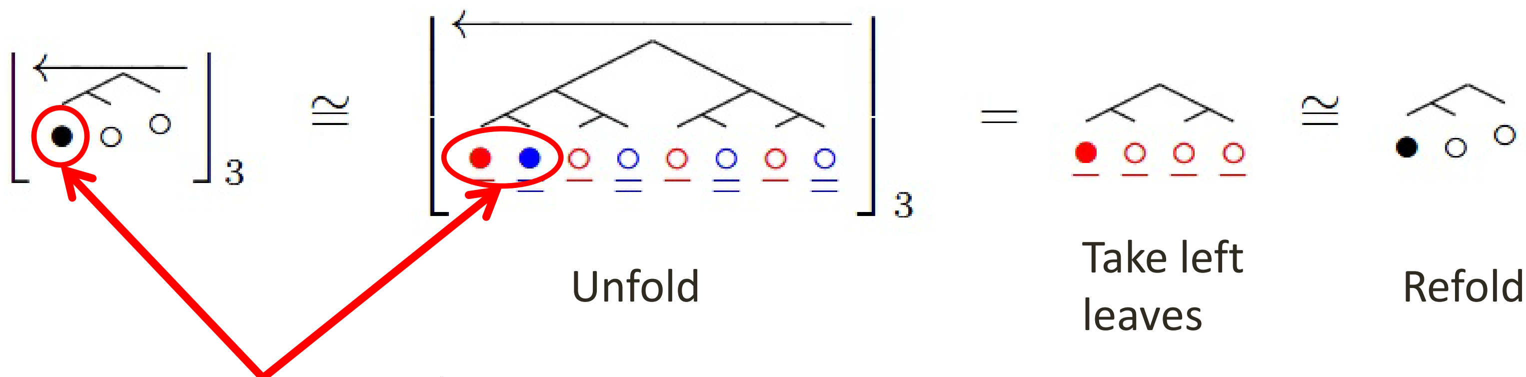

In general, $x \rightsquigarrow x \frown x$, and then we take the left/right, leaving x

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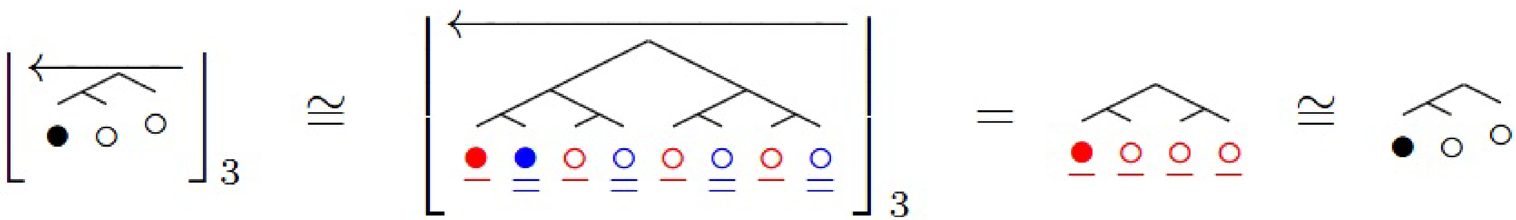

Unfold

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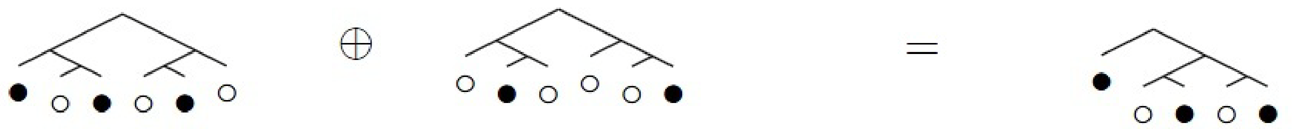
Refold

2. If $\tau_1 \oplus \tau_2 = \tau_3$, then $\lfloor \overleftarrow{\tau_1} \rfloor_n \oplus \lfloor \overleftarrow{\tau_2} \rfloor_n = \lfloor \overleftarrow{\tau_3} \rfloor_n$.

“Proof.”

Premise



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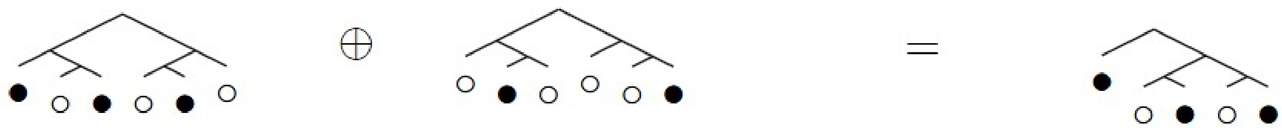
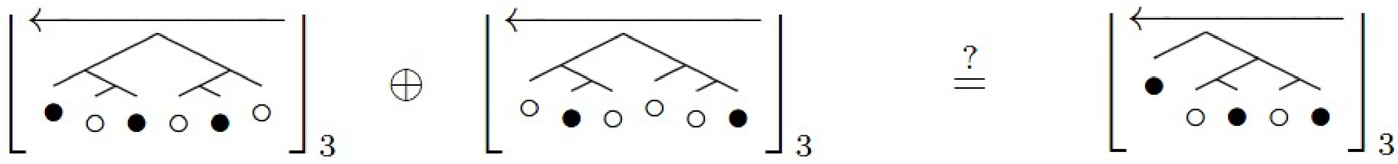


Apply round



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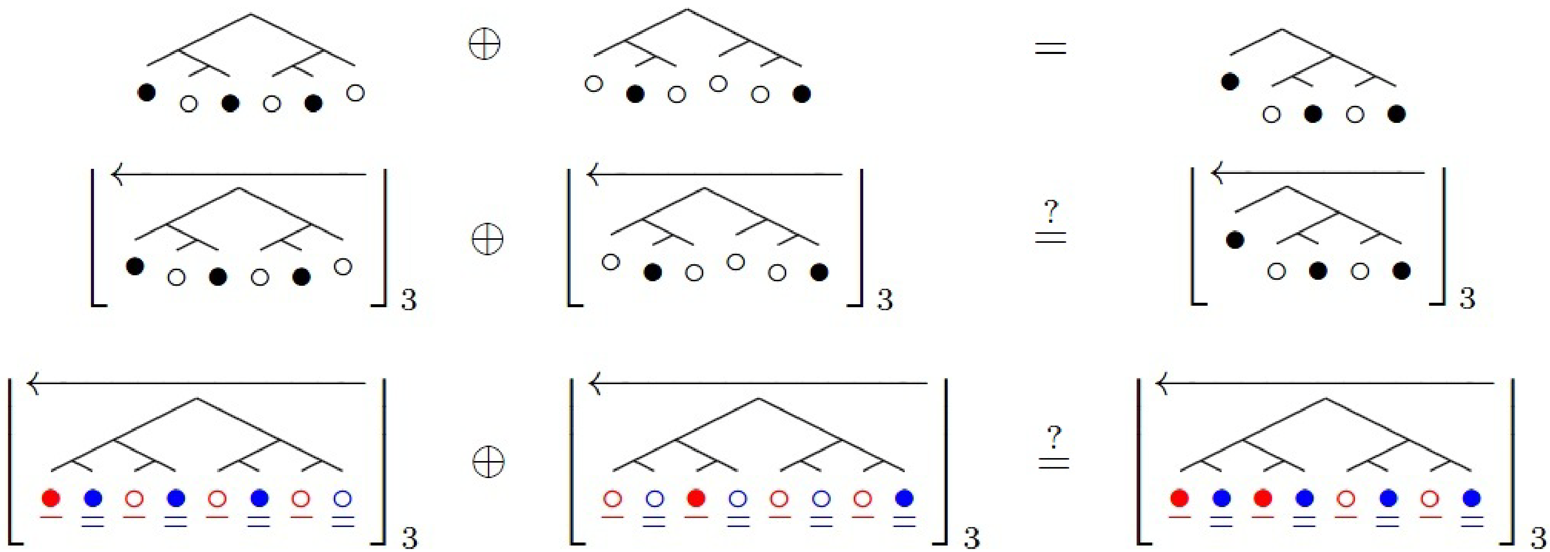

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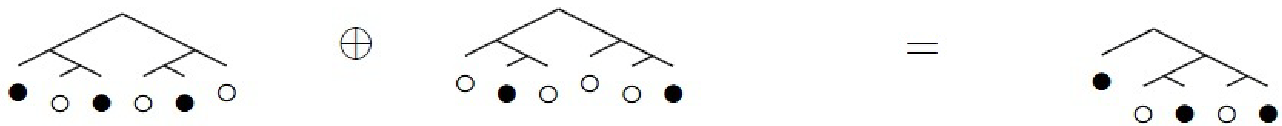

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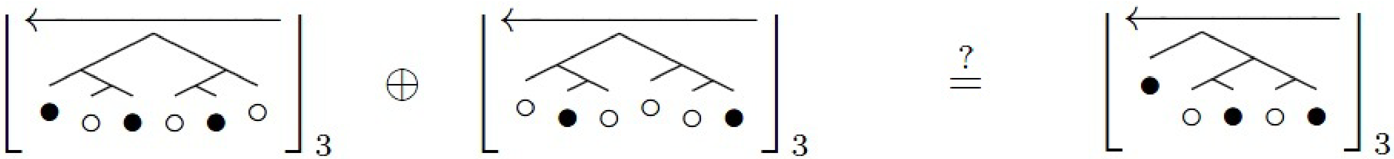

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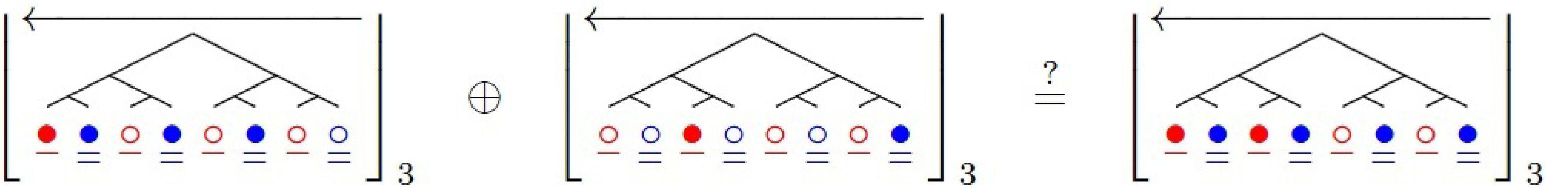
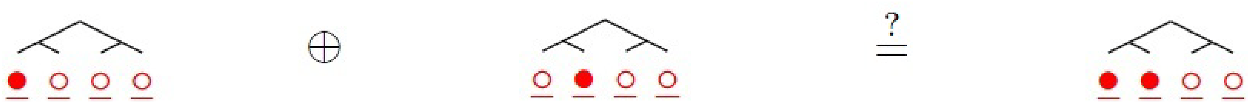


Take left leaves



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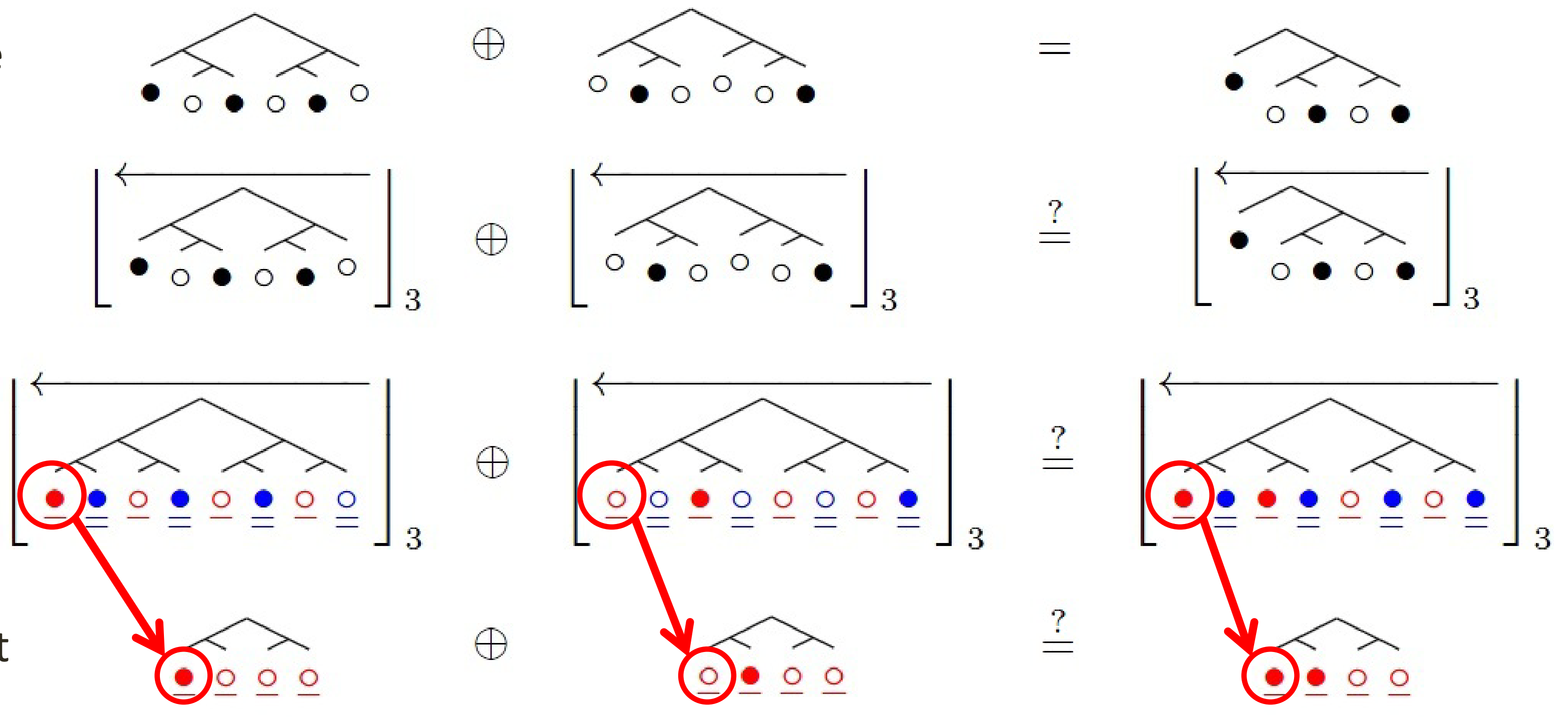

They joined before – joining occurs leafwise – so of course they join after!

“Proof.”

Premise

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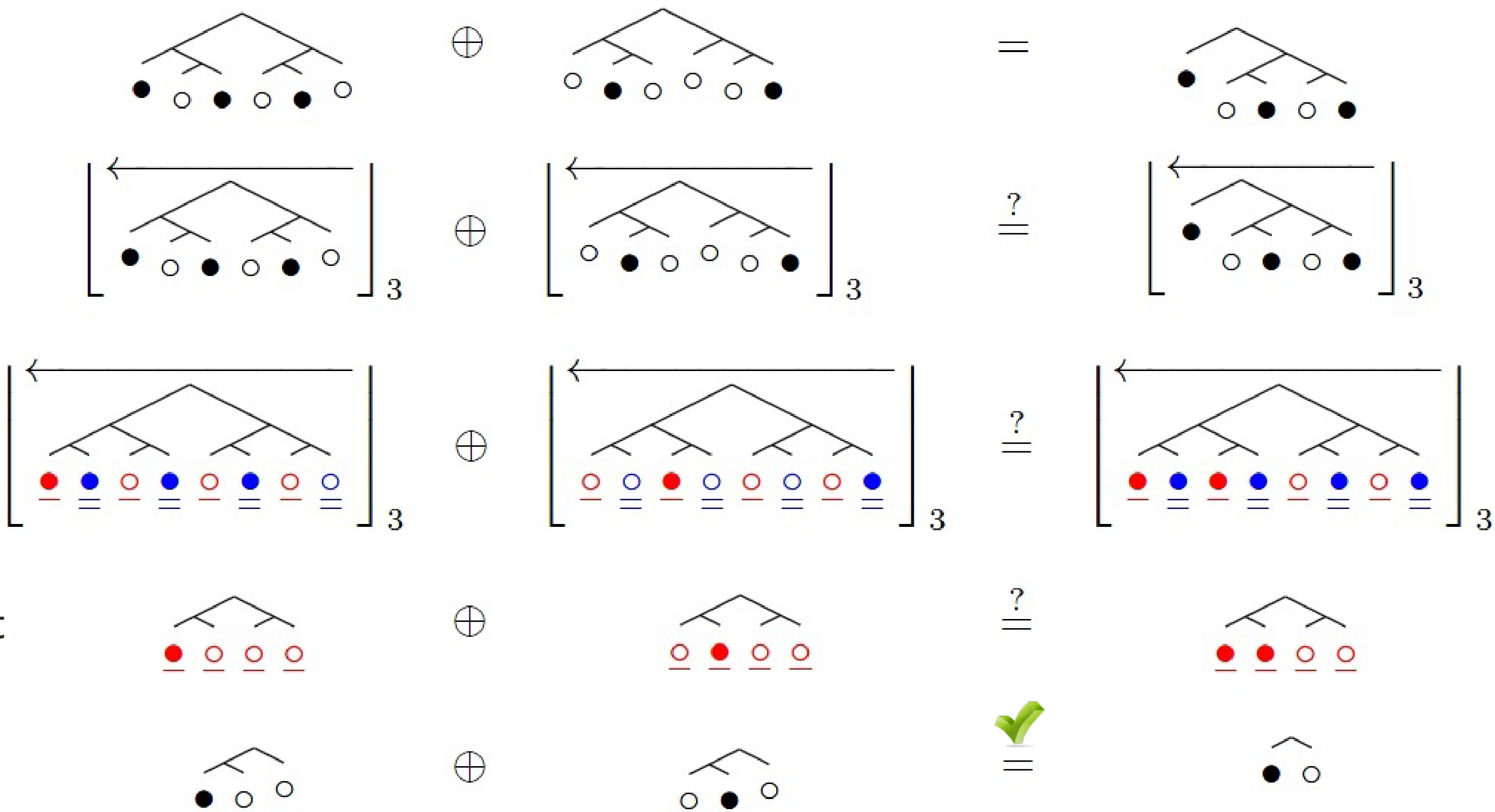
Unfold

Take left leaves

Refold



Proof sketch: finite SAT

- Theorem 1: finite search for SAT
  - Given Σ, $\exists\sigma.\ \sigma \vDash \Sigma$ iff $\exists\sigma.\ |\sigma|=|\Sigma| \wedge \sigma \vDash \Sigma$

- $\leftarrow$: trivial.

- $\rightarrow$: Take σ and repeatedly round it until it is of height $|\Sigma|$. Each equation in $|\Sigma|$ will still hold as long as we also round all constants (property 2), and since we are never rounding to height $|\Sigma|$, the constants in Σ are not changing (property 1), i.e., it is the same system of equations.

Completeness theorem (IMPL)

- Theorem 2: finite search for IMPL
  - Given Σ and Σ', $(\forall \sigma.\ \sigma \models \Sigma \rightarrow \sigma \models \Sigma')$ iff
    $(\forall \sigma.\ |\sigma| = |\Sigma| \rightarrow \sigma \models \Sigma \rightarrow \sigma \models \Sigma')$

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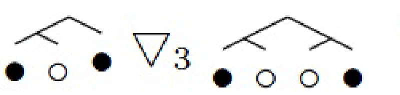
- Strategy:
  - Definition by example: averaging two trees
  - Proofs by example: properties of averaging
  - Proof sketch of main theorem

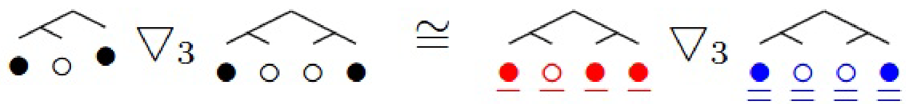
Averaging Trees

- Define $\tau_l \bigtriangledown_n \tau_r$ “averaging two trees at height n”:



Averaging Trees

- Define $\tau_l \ \nabla_n \ \tau_r$ "averaging two trees at height n":
  1. Unfold τ to height ***n*-1**



1. Unfold to **n-1**

Averaging Trees

- Define $\tau_l \bigtriangledown_n \tau_r$ "averaging two trees at height n":
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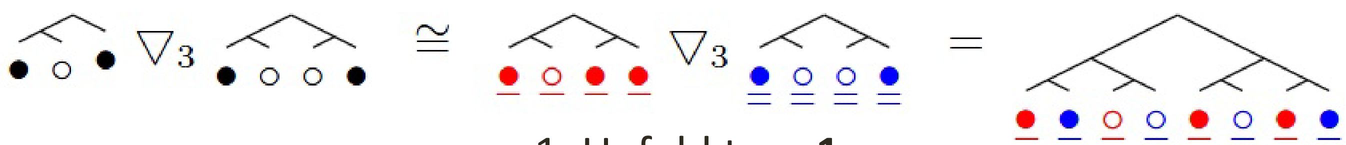
  2. Combine pairwise: left argument become left leaves in result; right argument become right leaves

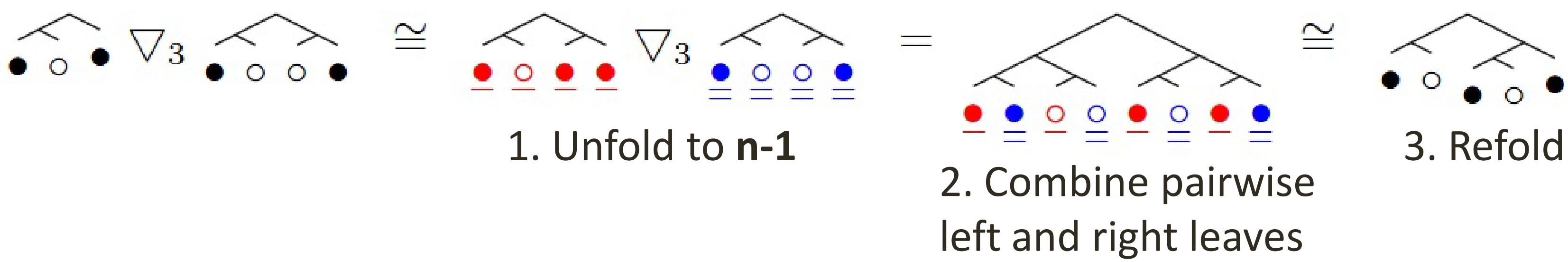


1. Unfold to **n-1**

2. Combine pairwise left and right leaves

Averaging Trees

- Define $\tau_l \ \nabla_n \ \tau_r$ "averaging two trees at height n":
  1. Unfold τ to height ***n*-1**
  2. Combine pairwise: left argument become left leaves in result; right argument become right leaves
  3. Refold as needed



Key properties of averaging

1. Averaging is the inverse of rounding, i.e.,

$$\lfloor \overleftarrow{\tau} \rfloor_n \ \nabla_n \ \lfloor \overrightarrow{\tau} \rfloor_n = \tau$$

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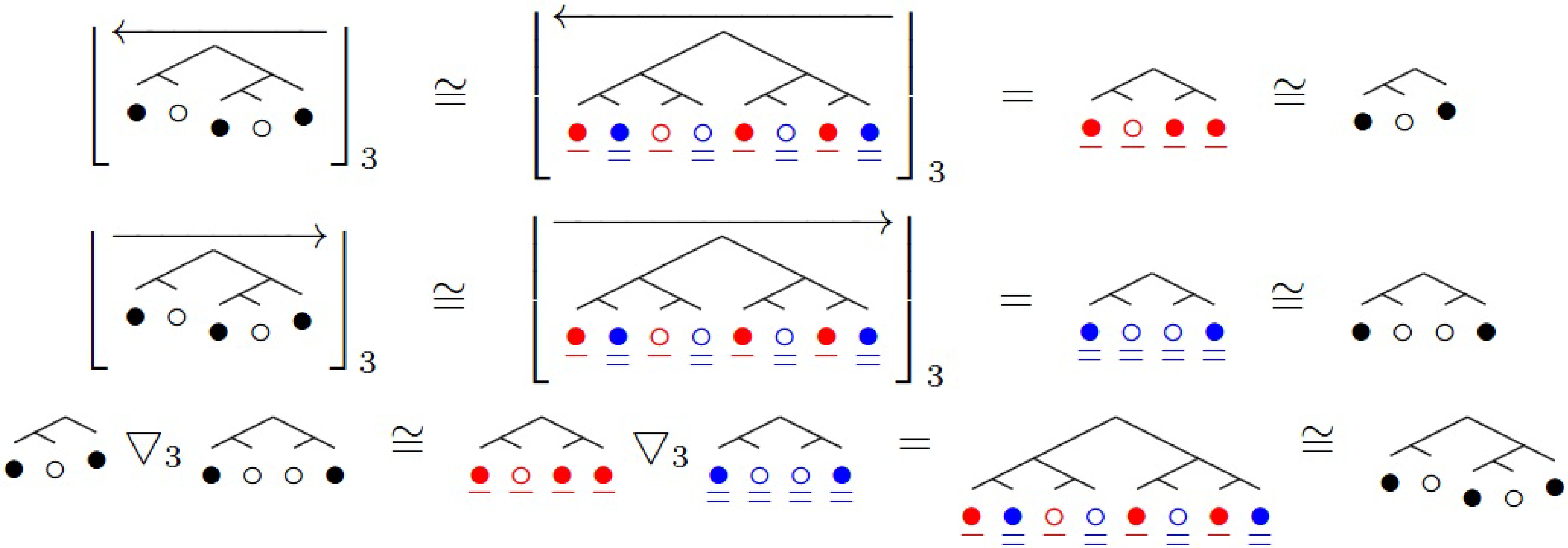
- “Proof.”



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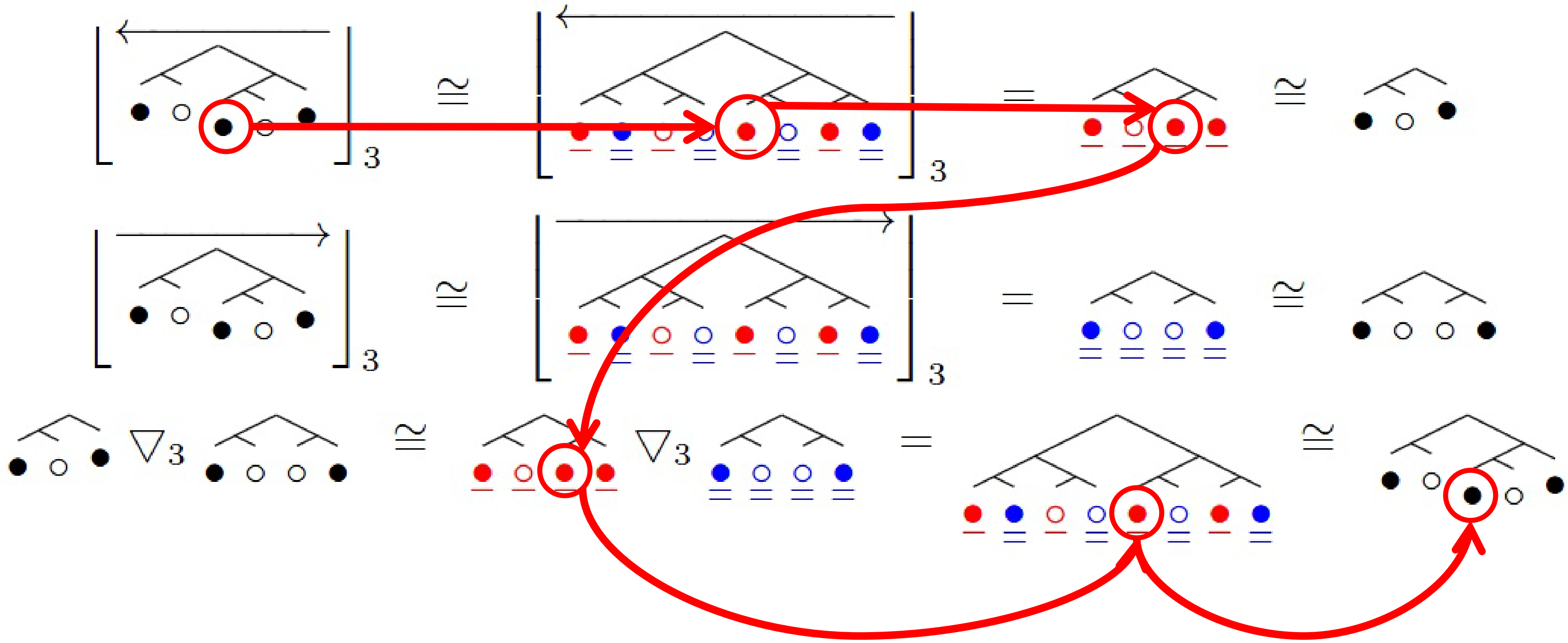

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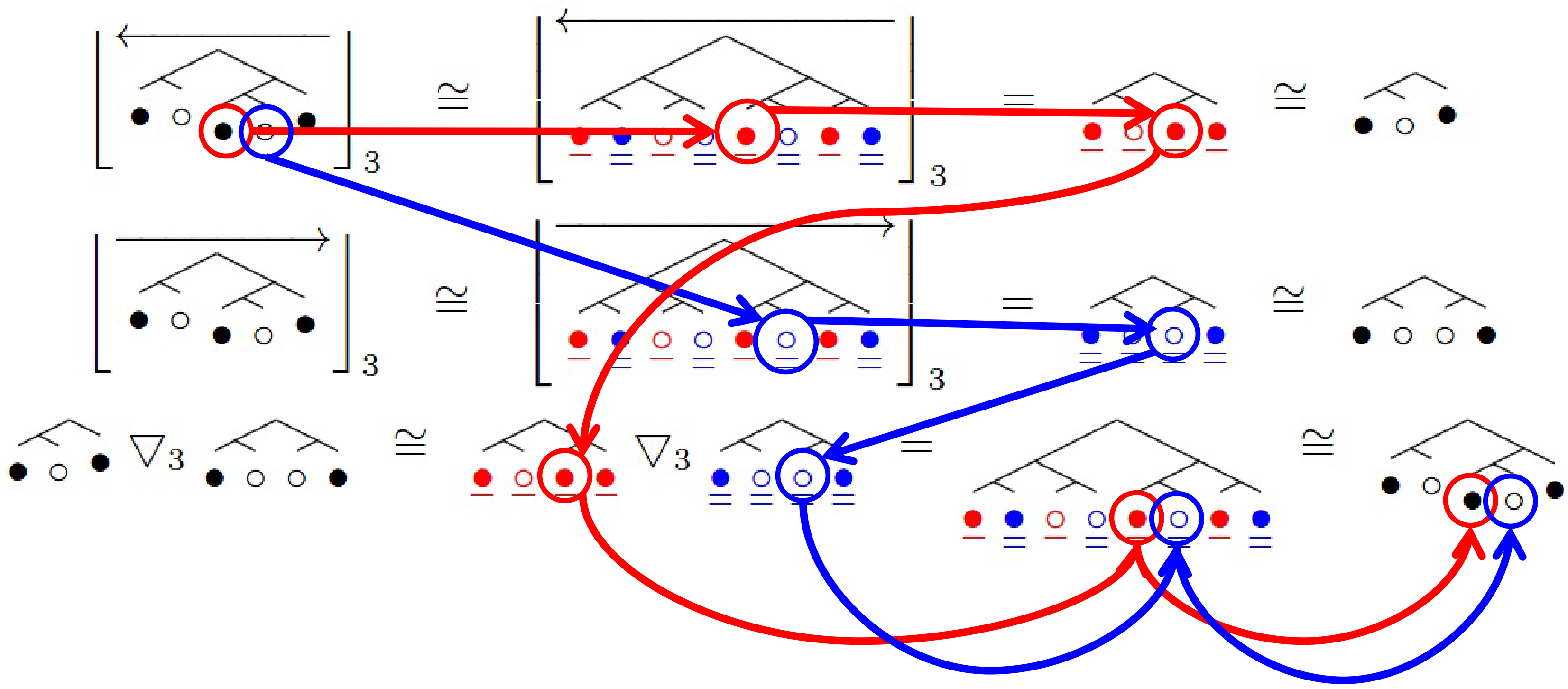

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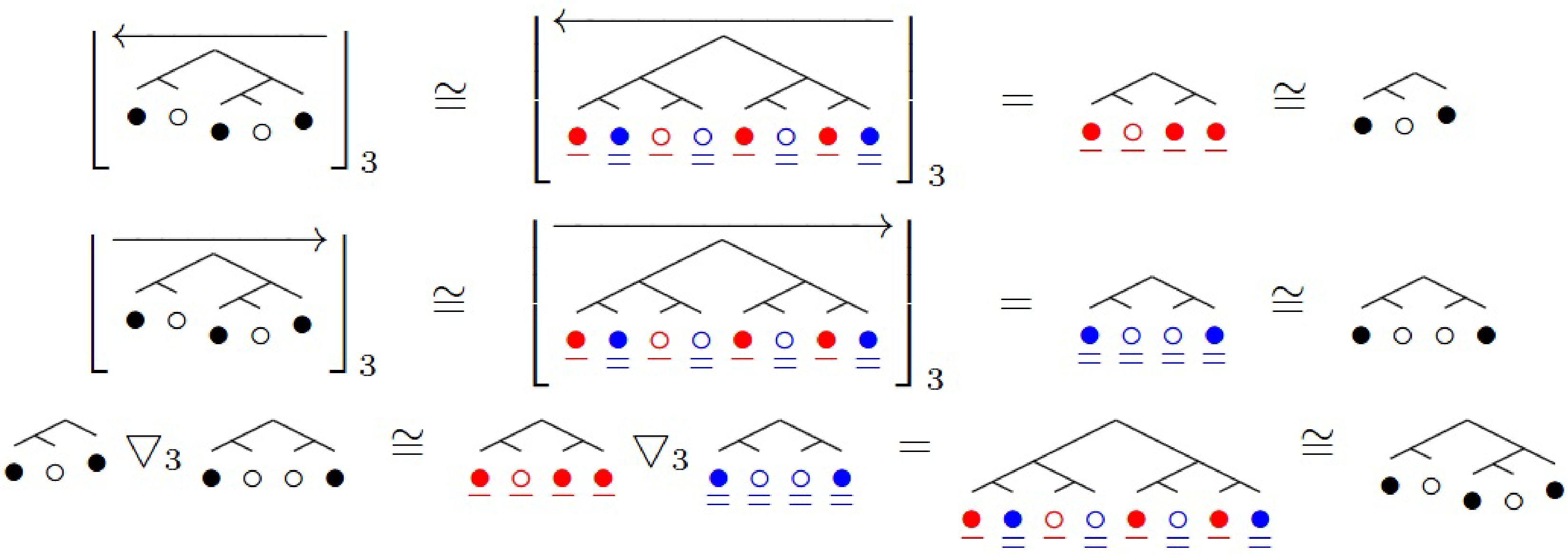
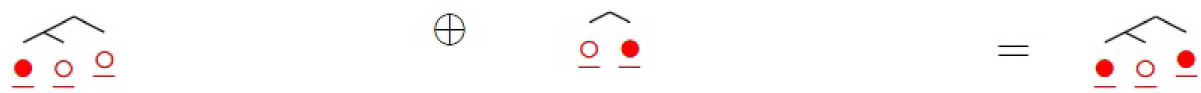
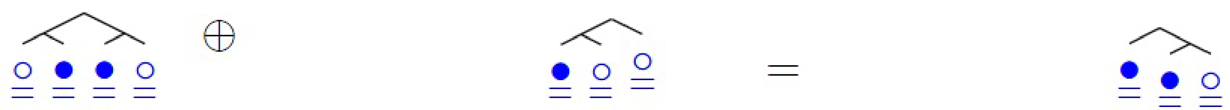


2. If $\tau_1 \oplus \tau_2 = \tau_3$ and $\tau_1' \oplus \tau_2' = \tau_3'$, then $(\tau_1 \ \nabla_n \ \tau_1') \oplus (\tau_2 \ \nabla_n \ \tau_2') = (\tau_3 \ \nabla_n \ \tau_3')$.

“Proof.”

Premise 

Premise 

“Proof.”

Premise

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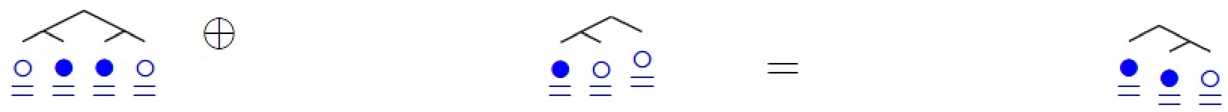
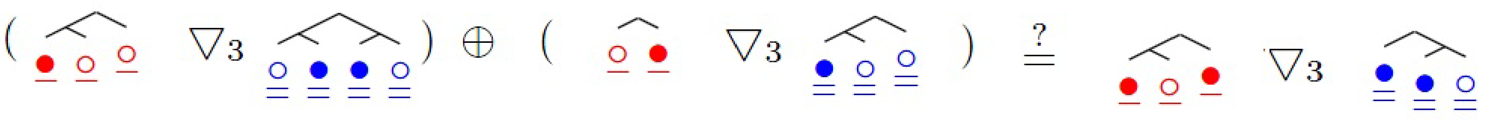


Apply average



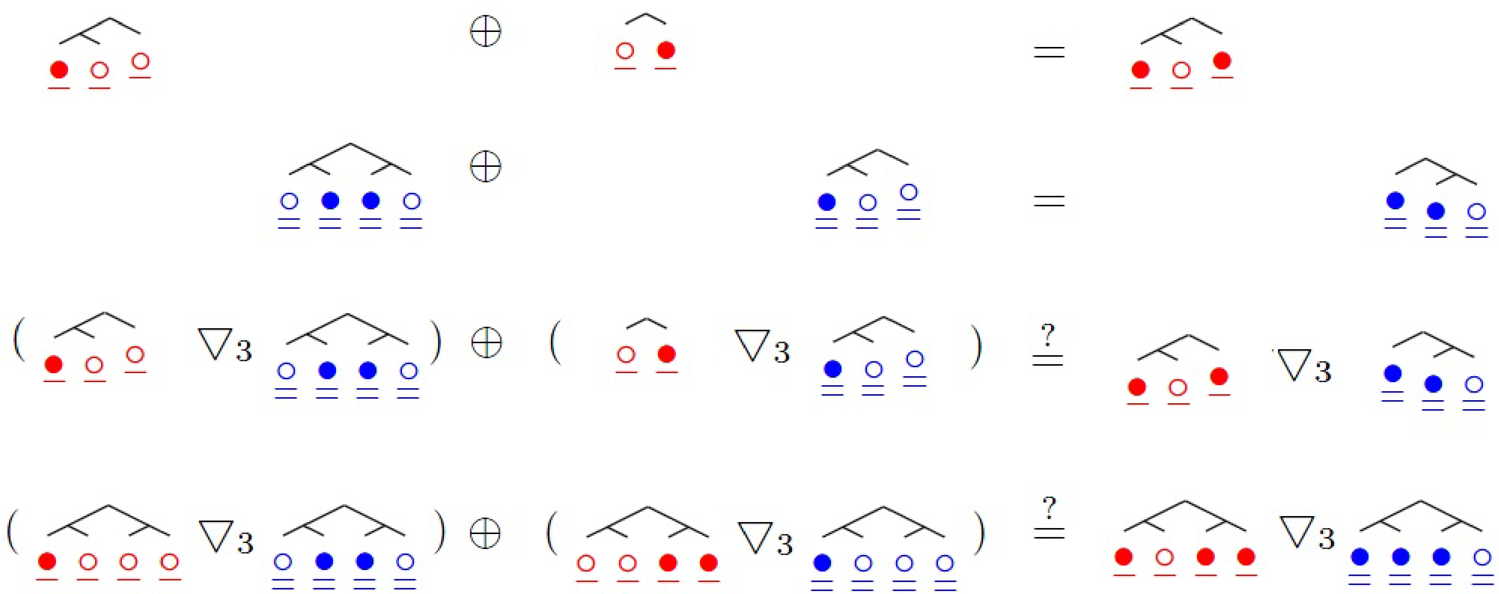
“Proof.”

Premise

Premise

Apply average

Unfold



“Proof.”

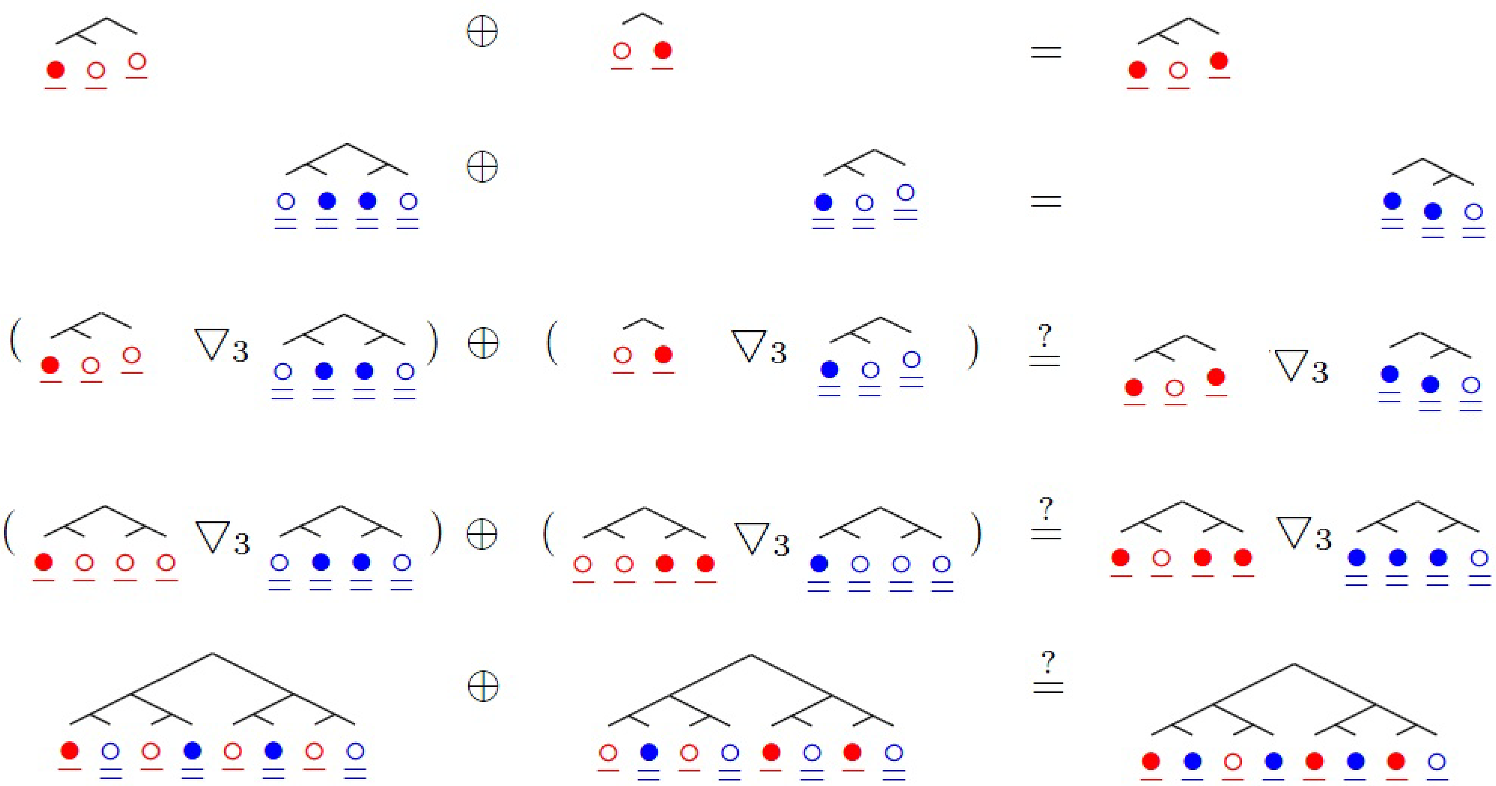
Premise

Premise

Apply average

Unfold

Calculate



“Proof.”

Premise

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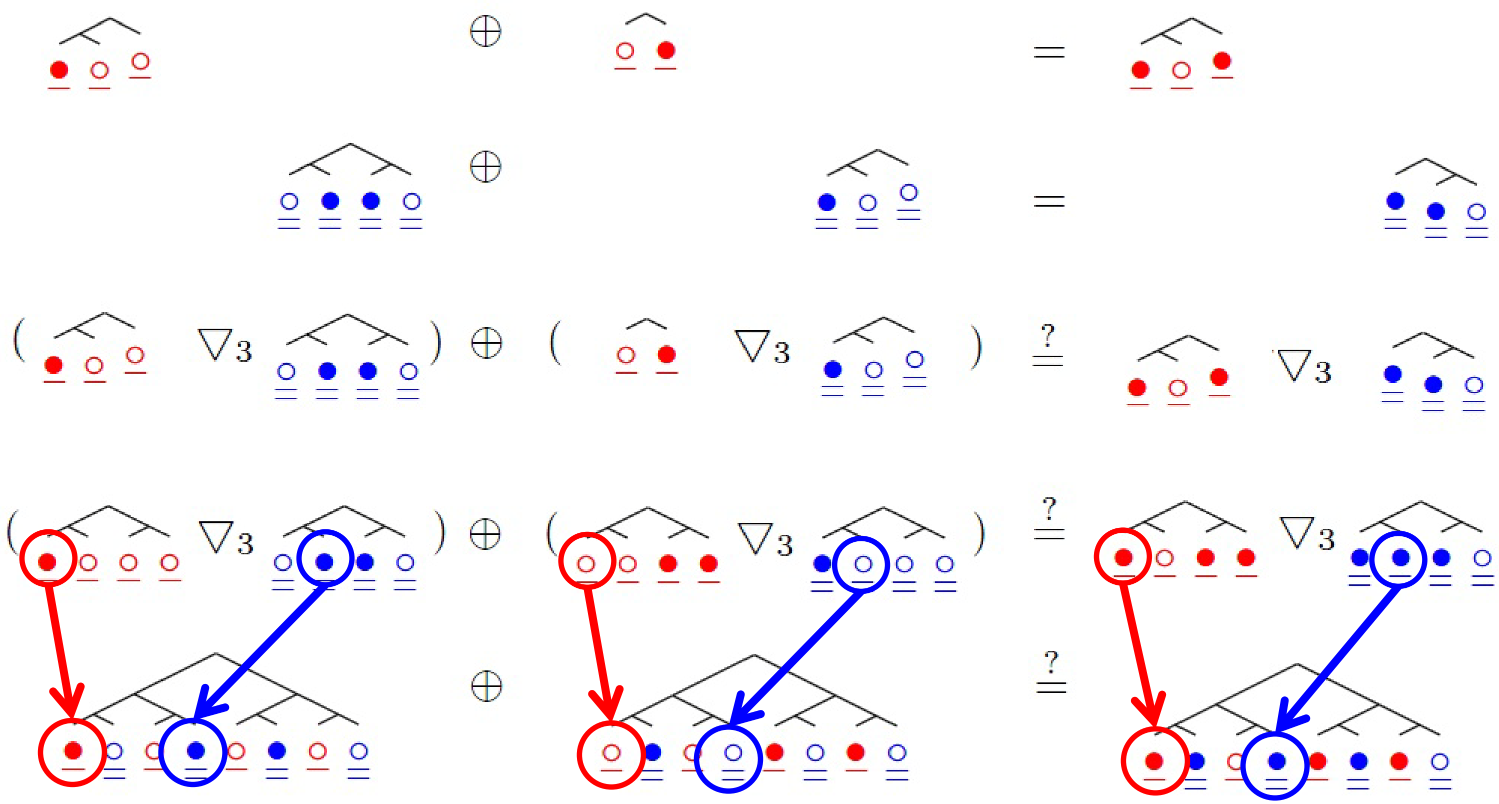


Again, because joining occurs leafwise, if they joined before they join after!

“Proof.”

Premise

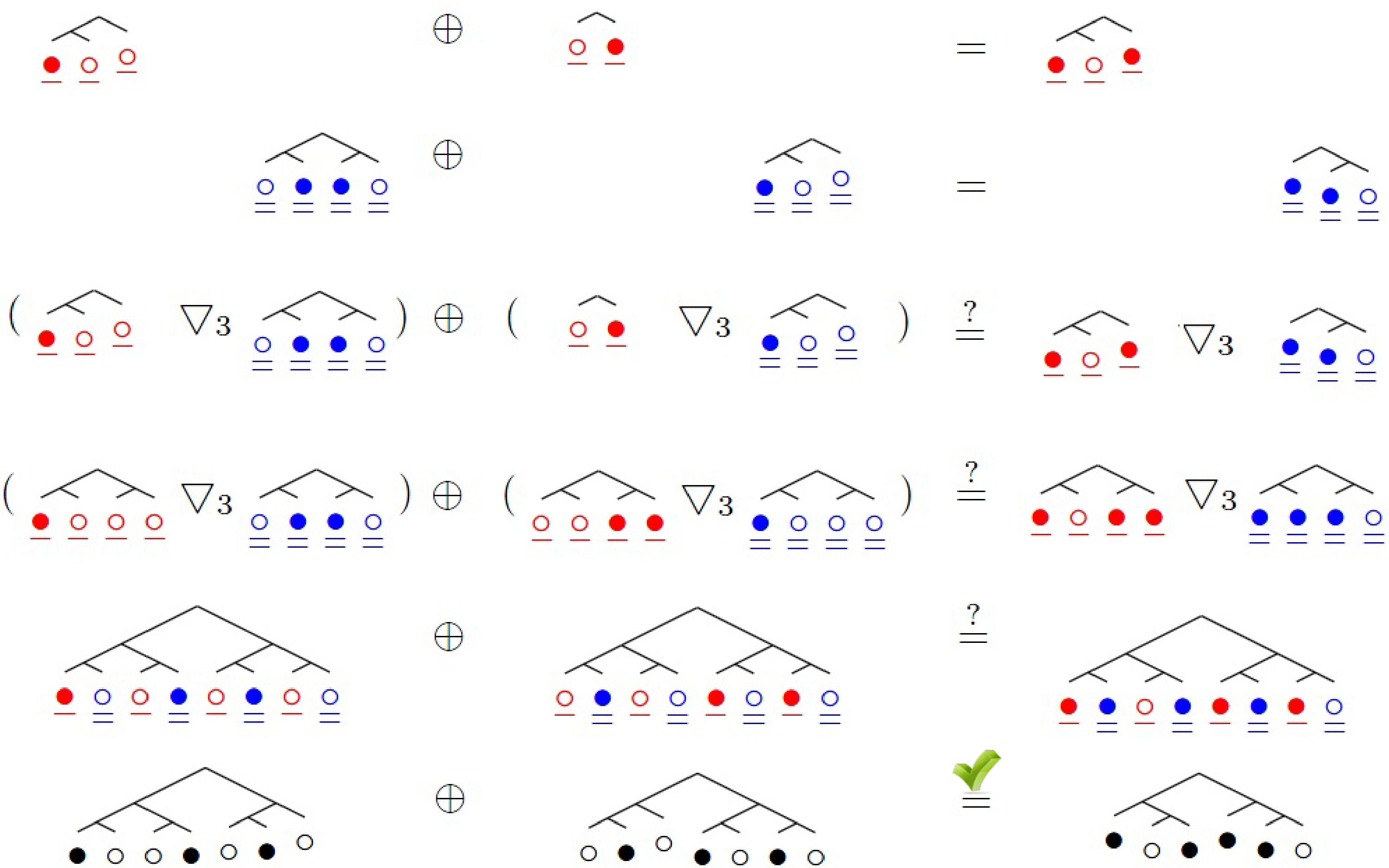
Premise

Apply average

Unfold

Calculate

Refold



Proof sketch: finite IMPL

- Theorem 2: finite search for IMPL
  - Given Σ and Σ', $(\forall\, \sigma.\ \sigma \models \Sigma \rightarrow \sigma \models \Sigma')$ iff
    $(\forall\, \sigma.\ |\sigma|=|\Sigma| \rightarrow \sigma \models \Sigma \rightarrow \sigma \models \Sigma')$
- $\rightarrow$: trivial.
- $\leftarrow$: Consider the case when $|\sigma|=|\Sigma|+1$. By the rounding lemmas, both the left round σ_l and right round σ_r of σ are still solutions for Σ (and have height $|\Sigma|$). Then we apply our hypothesis to learn that σ_l and σ_r are also solutions of Σ'. By averaging property 2, their average is a solution of Σ', and by averaging property 1, their average is equal to σ.

Plan of attack

1. Fractional Shares ✔
2. Verification Tools ✔
3. Our Decision Procedures ✔
4. Completeness ✔
5. Experimental Results

HIP/SLEEK Embedding

	SAT					IMPL				
test	call no.	BndP (ms)	ShP (ms)	SAT no.	SAT (ms)	call no.	BndP (ms)	ShP (ms)	SAT no.	SAT (ms)
barrier-weak	116	0.4	610	73	530	222	2.1	650	42	450
barrier-strong	116	0.6	660	73	510	222	2.2	788	42	460
barrier-paper	116	0.7	664	73	510	216	2.2	757	42	460
barrier-paper-ex	114	0.8	605	71	520	212	2.3	610	40	430
fractions	63	0.1	0.1	0	0	89	0.1	110	11	110
fractions1	11	0.1	0.1	0	0	15	0.1	31.3	3	30
barrier	68	0.1	0.9	0	0	174	1.2	3.9	0	0
barrier3	36	0.2	0.1	0	0	92	0.2	2.2	0	0
barrier4	59	0.1	0.7	0	0	140	0.9	2.4	0	0
read_ops	14	FAIL	210	14	208	27	FAIL	317	9	150
construct	4	FAIL	70	4	65	17	FAIL	880	17	270
join_ent	3	FAIL	70	3	30	3	FAIL	50	3	48

- Old tool is very fast…

HIP/SLEEK Embedding

	SAT					IMPL				
test	call no.	BndP (ms)	ShP (ms)	SAT no.	SAT (ms)	call no.	BndP (ms)	ShP (ms)	SAT no.	SAT (ms)
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- But it is incomplete… first two groups of tests were tweaked to avoid the (many) “dark zones”

HIP/SLEEK Embedding

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- New tool is slower, although **the rest of HIP/SLEEK takes more 3,000ms** on the first four tests

HIP/SLEEK Embedding

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- Most of the time is spent in the SMT solver (and communication/process overhead)

HIP/SLEEK Embedding

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- And, the new procedures are complete!

Standalone

- It’s actually really hard to develop tests to aggressive exercise the share procedures – in lots of code it will happen, but finding small examples is tricky.

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- We developed a standalone benchmark of 53 SAT and 50 IMPLY queries to stress the solver.
- Our new solver solved the entire suite in 1.4s.
- **Our old solver could solve fewer than 10%.**